1. (10%) A square matrix A is an idempotent matrix if $A^2 = A$. Find *all* the possible eigenvalues that a $n \times n$ idempotent matrix *can* have.

2. (10%) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. (a) (3%) Is A diagonalizable? (b) (7%) Please find A^{10}.

3. (10%) W is the solution set of the system of equations $x_1 - x_2 - 3x_3 = 0$ and $x_1 + x_2 + x_3 = 0$.

   Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$.Find the vector $\boldsymbol{w} \in W$ and $\boldsymbol{z} \in W^{\perp}$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$.

4. (10%) Given a $n \times n$ tridiagonal matrix A as below:

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

   Please find the determinant of A when $n = 2020$.

5. (10%)

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

   Please find the inverse of A (Hint: Does A have any property that can make the computation of its inverse simple?).

6. (10%) Let T be a linear operator on R^3 such that $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

   Please find the standard matrix of T.

7. (12%) Please find vector $\boldsymbol{z}$ such that $\|A\boldsymbol{z} - \boldsymbol{b}\|$ is a minimum, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

8. (10%) Consider the vector space of 2×2 real matrices. The inner product of matrix A and matrix B is defined as below:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

Please find the 2×2 real symmetric matrix that is closest to $\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$.

9. (18%) Let

$$A = \begin{bmatrix} 1 & 6 & -6 & -6 \\ 6 & 7 & -6 & -12 \\ 3 & 3 & -2 & -6 \\ 3 & 9 & -9 & -11 \end{bmatrix}$$

whose eigenvalues are -5, -2 and 1.

(a) (6%) Please find the eigenvalues of A^{-1}. (b) (6%) Please find the eigenvalues of $2A$. (c) (6%) Please find the eigenvalues of A^2.