第一大題選擇題：單選題與複選題之混合式試題，
考生應作答於『答案卡』（請勿作答於試卷之選擇題作答區）

1. (10%) Given an RLC circuit. The current i(t) can be expressed by $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=E(t)=0$ for $t>0$

where $R=1, C=\frac{2}{5}, L=\frac{1}{2}, i(0)=0, \frac{di(0)}{dt}=-8, \int_{-\infty}^{0} i(t)dt=\frac{8}{5}$

Which of the following are correct?

(A) i(t) approaches zero at $t\to\infty$ for any positive R, L ,C value,

(B) For t>0, i(t) does not have sinusoidal component,

(C) $i(1) > e^{-2}$,

(D) For t>0, i(t) is proportional to e^{-t},

(E) $i(\pi)=0$.

2. (5%) Solve the differential equation below.

$$y''+2y'+y=e^{-x}$$

Which of the following are the possible solutions?

(A) $y(x)=e^{-x}$,

(B) $y(x)=(5+\frac{1}{4}x^2)xe^{-x}$,

(C) $y(x)=e^{-x}+2xe^{-x}$,

(D) $y(x)=(2+\frac{1}{2}x)xe^{-x}$,

(E) $y(x)=e^{-x}+2x^2e^{-x}$.

3. (5%) For the differential equation below.

$$y^{(4)}+2y''+y=f(x)$$

Which of the following are correct?

(A) General solution form include $y(x)=x\cdot\cos(x)$,

(B) General solution form include $y(x)=e^{x}$,

(C) General solution form include $y(x)=\sin(x)$ and $y(x)=x\cdot\sin(x)$,

(D) If $f(x)=3e^{x}$, particular solution is in the form of c_1e^{x},

(E) If $f(x)=3\cos(x)$, particular solution is in the form of $c_1\cos(x)+c_2\sin(x)$.

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4. (5%) A heart pacemaker consists of a switch, a battery, a capacitor with constant capacitance C, and the heart as a resister with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage y(t) across the heart satisfies the linear differential equation :

$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=t$, where t is the time variable. Assume that initially, i.e., at t = 0, the voltage y(t) across the heart is equal to 4 and the constant $RC=1$. Which of the following statements are true?

(A) For $t\geq 0$, the particular solution of the linear differential equation is $y_p(t)=t-1$,

(B) For $t\geq 0$, the particular solution of the linear differential equation is $y_p(t)=4(t-1)$,

(C) For $t\geq 0$, the particular solution of the linear differential equation is $y_p(t)=ce^{-t}$, c is any constant,

(D) For $t\geq 0$, the solution of the linear differential equation is $y(t)=t-1+5e^{-t}$,

(E) For $t\geq 0$, the solution of the linear differential equation is $y(t)=4(t-1)+8e^{-t}$.

5. (5%) Consider one specially designed circuit with one capacitor, one inductor, and four resisters. The output voltage of the capacitor is denoted as x(t) and the output current of the inductor is denoted as y(t). Assume that initially x(0)=0 and y(0)=0 and these two devices might interact with each other by the following behavior. For the capacitor, the changing rate of x(t) is declined at a rate of -3x(t), and simultaneously increased at a rate of y(t), and also positively depends on the independent current source $g(t)=3t$. Similarly, for the inductor, the changing rate of y(t) declines at a rate of -4y(t), and simultaneously increases at a rate of 2x(t), and also positively depends on another independent voltage source $h(t)=e^{-t}$. Which of the following statements are true?

(A) $\frac{dx(t)}{dt}=-3x(t)+y(t)+3t$, (B) $\frac{dx(t)}{dt}=-3y(t)+x(t)+3t$, (C) $\frac{dx(t)}{dt}=-3x(t)+y(t)+e^{-t}$,

(D) $\frac{dy(t)}{dt}=2x(t)-4y(t)+e^{-t}$, (E) $\frac{dy(t)}{dt}=2x(t)-4y(t)+3t$.

6. (5%) A forced, undamped, and resonant motion of a mass on a spring can be described in the following equation: $\dfrac{d^2x(t)}{dt^2}+16x(t)=\cos(4t)$, where x(t) is the location of the mass. In the beginning, the mass is located at $x(0)=0$ and the initial velocity of the mass is equal to 1, i.e., $\dfrac{dx(0)}{dt}=1$. The Laplace transform X(s) of x(t) is:

(A) $X(s)=\dfrac{2s+16}{(s^2+16)^2}$, (B) $X(s)=\dfrac{s^2+s+16}{(s^2+16)^2}$, (C) $X(s)=\dfrac{1}{s^2+16}+\dfrac{s}{(s^2+16)^2}$,

(D) $X(s)=\dfrac{1}{s+16}+\dfrac{s}{(s+16)^2}$, (E) $X(s)=\dfrac{1}{s+16}+\dfrac{s}{s^2+16}$.

7. (5%) A semi-infinite plate coincides with the region defined by $0\le x\le\pi, y\ge 0$. The left end is held at temperature: exp(−y), and the right end is held at temperature zero for $y\ge 0$. The bottom of the plate is insulated.

$$u(x,y)=\frac{2}{\pi}\int_0^\infty \frac{\sinh[f(x)]}{g(\alpha)\sinh(\alpha\pi)}p(y)d\alpha$$

(A) $g(\alpha)=1+\alpha$, (B) $f(x)=\alpha x$, (C) $f(x)=\alpha(\pi-x)$, (D) $p(y)=\sin(\alpha y)$,
(E) $p(y)=\cos(\alpha y)$.

8. (5%) A string is stretched and secured on the x-axis at $x=0$ and $x=1$ for $t>0$, that is initially held at these points $0.01\sin(3\pi x)$ and then simultaneously released at all points at time $t=0$. The string is released from rest from the initial displacement. (Wave equation constant: a) $u(x,t)=0.01f(x)g(t)$

(A) $g(t)=\cos(\pi at)$, (B) $f(x)=\sin(3\pi x)$, (C) $f(x)=\sin(3\pi ax)$, (D) $g(t)=\cos(3\pi at)$,
(E) $g(t)=\sin(3\pi at)$.

9. (5%) An undamped string/mass system, in which the mass $m=1$ slug and the spring constant $k=10$ lb/ft, is driven by the 2-periodic external force $f(t)=1-t, 0<t<2;\ f(t+2)=f(t)$. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is odd. Find a particular solution Xp(t).

$$Xp(t)=\sum_{n=1}^{\infty}\frac{2}{s(10-k)}\sin[p(t)]$$

(A) $p(t)=n\pi t$, (B) $s=n\pi$, (C) $k=n\pi$, (D) $k=n^2\pi^2$, (E) $s=n^2\pi^2$.

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10. (5%) Please pick the statements below that are true.

(a) The rank of the coefficient matrix of a consistent system of linear equations is equal to the number of basic variables in the general solution of the system.

(b) The matrix-vector product of an $m \times n$ matrix A and a vector in $\mathcal{R}^n$ is a linear combination of the columns of A.

(c) A subset of $\mathcal{R}^n$ containing fewer than n vectors must be linearly independent.

(d) If S_1 and S_2 are finite subsets of $\mathcal{R}^n$ having equal spans, then S_1 and S_2 contain the same number of vectors.

(e) If the columns of an $n \times n$ matrix A for a generating set of $\mathcal{R}^n$, then the reduced row echelon form of A is I_n.

11. (5%) Please pick the statements below that are true.

(a) If A and B are $m \times n$ matrices and C is an $n \times p$ matrix, then $(A+B)C = AC + BC$.

(b) For any matrices A and B, if A is the inverse of B^T, then A is the transpose of B^{-1}.

(c) An $n \times n$ matrix is invertible if and only if its rows are linearly independent.

(d) The codomain of any function is contained in its range.

(e) If $f : \mathcal{R}^n \to \mathcal{R}^m$ and $g : \mathcal{R}^n \to \mathcal{R}^m$ are functions such that $f(\mathbf{e}_i) = g(\mathbf{e}_i)$ for every standard vector $\mathbf{e}_i$, then $f(\mathbf{v}) = g(\mathbf{v})$ for every $\mathbf{v}$ in $\mathcal{R}^n$.

12. (5%) Please pick the statements below that are true.

(a) A function is onto if its range equals its domain.

(b) A linear transformation is one-to-one if and only if every vector in its range is the image of a unique vector in its domain.

(c) For any $m \times n$ matrices A and B and any scalars c and d, $(cA + dB)^T = cA^T + dB^T$.

(d) No scaling operations are required in the forward pass of Gaussian elimination.

(e) If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible.

13. (5%) Consider the matrix $A = \begin{bmatrix} c & -3 \\ -5 & c+2 \end{bmatrix}$. Which of the following values of c will make the matrix A invertible?
(a) 3; (b) 4; (c) 5; (d) 6; (e) None of the above.

14. (5%) Which of the following subsets of $\mathcal{R}^n$ is a subspace?

(a) $\left\{ \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \in \mathcal{R}^2 \;\middle|\; 2u_1^2 + 3u_2^2 \leq 12 \right\}$.

(b) $\left\{ \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \in \mathcal{R}^3 \mid 2u_1 + 5u_2 - 4u_3 = 0 \right\}$.

(c) $\left\{ \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \in \mathcal{R}^3 \mid u_1 u_2 = u_3^2 \right\}$.

(d) $\left\{ \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \in \mathcal{R}^3 \mid 2u_1 + 5u_2 - 4u_3 = 0 \text{ and } u_1 - 2u_2 + 3u_3 = 0 \right\}$.

(e) $\left\{ \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \in \mathcal{R}^3 \mid 2u_1 + 5u_2 - 4u_3 = 0 \text{ or } u_1 - 2u_2 + 3u_3 = 0 \right\}$.

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15. (5%) Which of the following statements are correct?

   (a) A basis for a subspace is a linearly independent subset of the subspace that is as large as possible.

   (b) If V is a subspace of dimension k, then every generating set for V contains exactly k vectors.

   (c) If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is any basis for $\mathcal{R}^n$, then there exists a vector $\mathbf{v} \in \mathcal{R}^n$ so that $\mathbf{v}$ can not be expressed as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$.

   (d) Let T be a linear operator on $\mathcal{R}^n$, and $\mathcal{U}$ and $\mathcal{V}$ be two bases for $\mathcal{R}^n$. Denote $[T]_{\mathcal{B}}$ as the representative matrix of T with respect to any basis $\mathcal{B}$ for $\mathcal{R}^n$. Then, $[T]_{\mathcal{U}}$ and $[T]_{\mathcal{V}}$ are similar.

   (e) If $\mathcal{B}$ is a basis for $\mathcal{R}^n$ and T is the identity operator on $\mathcal{R}^n$, then $[T]_{\mathcal{B}} = I_n$.

16. (5%) Which of the following statements are correct?

   (a) If λ is an eigenvalue of an $n \times n$ matrix A, then for any $\mathbf{v} \in \mathcal{R}^n$, $A\mathbf{v} = \lambda\mathbf{v}$.

   (b) If there exists some $\mathbf{v} \in \mathcal{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$, then λ is an eigenvalue of A.

   (c) If two matrices have the same characteristic polynomials, then they have the same eigenvalues.

   (d) A diagonal matrix is always diagonalizable.

   (e) Let A and P be $n \times n$ matrices. If the columns of P form a set of n linearly independent eigenvectors of A, then $P^{-1}AP$ is a diagonal matrix.

第二大題：非選擇題

1. (15%) You did an experiment and got the following data points of (x, y) pair:

$$(0,0), (2,2), (3,6), \text{ and } (4,12).$$

You hypothesize that $y = f(x) = a + bx$, where $a, b \in \mathcal{R}^1$. You would like to fit $f(x)$ to the data points.

(1.1) Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and we express the fitting of $f(x)$ to the four data points into a matrix form $A\mathbf{v} = \mathbf{c}$. $A =$? and $\mathbf{c} =$? (3%)

(1.2) Find a column span of matrix A, that is, $CS(A) =$? (3%)

(1.3) Now you want to perform a least-square-error fit (also called a linear regression) of $f(x)$ to the four given data points, that is,

$$\min_{\mathbf{v}} ||A\mathbf{v} - \mathbf{c}||^2.$$

Please explain why the optimal solution, $\mathbf{v}^*$, is the projection of vector $\mathbf{c}$ onto $CS(A)$. (4%)

(1.4) Derive the projection matrix that projects vector $\mathbf{c}$ onto $CS(A)$. (5%)