1. Consider the shortest paths on planar integer grid lines going from $(0,0)$ to (m,n) (each step either increases the x value by 1 or increases the y value by 1).
   Find
   (a) (5 points) The number of shortest paths that avoids (x,y), $1 < x < m$ and $1 < y < n$.
   (b) (10 points) The number of shortest paths that avoids both (x_1,y_1) and (x_2,y_2), $1 < x_1 < x_2 < m$ and $1 < y_1 < y_2 < n$.

2. (15 points) Prove or disprove that for every positive integer n,

$$(-1)^n C(2n,n) = \sum_{k=0}^{2n}(-1)^k[C(2n,k)^2],$$

   where $C(n,k)$ is the coefficient of the x^k term in the expansion of $(1+x)^n$
   (Hint: $(x^2-1)^{2n} = (x+1)^{2n}(x-1)^{2n}$)

3. (15 points) Solve the following recurrence:

$$a_1 = 8, \quad (1)$$
$$a_2 = 8, \quad (2)$$
$$a_n = a_{n-1}a_{n-2}^2, \text{ for all } n \geq 3. \quad (3)$$

4. (25 points) For each of the following statements, determine whether it is true of false. Prove the correctness of your answers. Answers without explanations will not receive any credits.
   (a) Every propositional logic statement has an equivalent statement using only $\vee$ and $\neg$.
   (b) Every propositional logic statement has an equivalent statement using only $\vee$ and $\wedge$.
   (c) If a relation R is symmetric and transitive, then it must be reflexive.
   (d) If a relation R is reflexive and transitive, then it must be symmetric.
   (e) Given a set S. If S is countably infinite, then 2^S must not be countably infinite.

5. (15 points) Given a directed graph G. Suppose that for any two vertices u, v, either $(u,v) \in E$ or $(v,u) \in E$ but not both. If G has a cycle, prove that G has a cycle of length 3.

6. (15 points) Given an undirected graph G with maximum vertex degree d. Prove that the chromatic number of G is at most $d+1$.