※ 注意：請用 2B 鉛筆作答於答案卡，並先詳閱答案卡上之「畫記說明」。

1. (5%) A function $y(x)$ satisfies the following
$$(y')^2 = 9x^4 y, \qquad y(0) = 0.$$
Find the possible values of y(2) from below
(A) 0
(B) -4
(C) 16
(D) -64
(E) None of the above

2. (5%) For the following linear differential equation
$$x^3 y''' + 2x^2 y'' + xy' = y, \qquad x > 0.$$
Choose from the below set of independent solutions of the above equation.
(A) $\{x, x\sin x, x\cos x\}$
(B) $\{x, x\sin 3x\}$
(C) $\{x, \sin\ln x\}$
(D) $\{x, \sin\ln 3x\}$
(E) None of the above

3. (5%) Consider the following differential equation of a function y(x)
$$y^{(4)} + 4y = 0.$$
Choose from the below possible solutions of the above equation.
(A) $y(x) = e^{2x}$
(B) $y(x) = e^{-x}\cos x$
(C) $y(x) = e^{-\sqrt{2}x}\sin x$
(D) $y(x) = e^{x}\cos(x + \pi/4)$
(E) None of the above

4. (5%) A function y(x) satisfies the following:
$$y'' + 4y' + 4y = 4e^{-2x}$$
Choose from the below possible solutions of the above equation.
(A) $y(x) = (x + 2x^2)e^{-2x}$
(B) $y(x) = e^{-2x}$
(C) $y(x) = (1 + 2x + x^2 + x^3)e^{-2x}$
(D) $y(x) = (1 + x)(1 + 2x)e^{-2x}$
(E) None of the above

5. (5%) A dynamical system with a time-varying state variable x(t) is described by the following differential equation
$$(t^2 + t - 2)x'' - 2(2t + 1)x' + 6x = 0, \qquad t \geq 0$$
Under the conditions
$$0 < x(1) < 1, x'''(1) = 0$$
Find the possible values of x(2) from below,
(A) 1
(B) 2
(C) 4
(D) 8
(E) None of the above

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6. (5%) Which of the following function is of exponential order?
   (A) $5e^{4t}$
   (B) $\cosh 3t$
   (C) $\cosh t^4$
   (D) $\sin e^{t^2}$
   (E) None of the above

7. (5%) Evaluate the Laplace or inverse Laplace transform of the following functions. Which are correct?
   (A) $f(t) = e^{-t} \times H(t-5),\ L\{f(t)\} = (1-5e^{-t})H(t-5)$
   (B) $f(t) = \int_0^5 H(\tau - t)d\tau,\ L\{f(t)\} = (5-t)H(5-t)$
   (C) $f(s) = \frac{1}{s}\tanh s,\ L^{-1}\{f(s)\} = H(t) - 2H(t-1) + 2H(t-3) - \cdots$
   (D) $f(s) = \frac{e^{-s}}{s^2+s+1},\ L^{-1}\{f(s)\} = \frac{2}{\sqrt{3}}H(t-1)e^{1-t/2}\sin 2\sqrt{3}(t-1)$
   (E) None of the above

8. (5%) The inverse Laplace transform of function $f(s) = \ln\left(\frac{s^2+1}{s^2+s}\right)$ is expressed as

$$f(t) = \frac{A(t)\cos t + B(t) + e^{C(t)}}{D(t)}$$

   Which of the followings are correct?
   (A) $A(t) = -2t$
   (B) $B(t) = 1$
   (C) $C(t) = -t$
   (D) $D(t) = \sin t$
   (E) None of the above

9. (5%) Given a forcing function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, 2 \le t < 3, 4 \le t < 5, \cdots \\ 0, & 1 \le t < 2, 3 \le t < 4, 5 \le t < 6, \cdots \end{cases}$$

   Solve the differential equation $x'' + x = f(t)$, with that $x(0) = x'(0) = 1$, and we have

$$x(t) = \sin A(t) + B(t)\cos C(t) + 2t + D(t)H(t-2) + E(t)H(t-4) + \cdots$$

   Which of the followings are correct?
   (A) $A(t) = t$
   (B) $B(t) = -1$
   (C) $D(t) = 4(\cos(2-t) - 1)$
   (D) $x(5) > 4$
   (E) None of the above

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10. (5%) Evaluate the Fourier or inverse Fourier transform of the following functions. Which are correct?

(A) $f(\omega) = H(\omega + a) - H(\omega - a), a > 0,\ F^{-1}\{f(\omega)\} = \sin x/ax$

(B) $f(x) = 4x^2 e^{-3|x|},\ F\{f(x)\} = 48\left[\frac{1}{(\omega^2+9)^2} - \frac{4\omega^2}{(\omega^2+9)^3}\right]$

(C) $f(\omega) = \frac{4\sin\omega}{\omega} - \frac{1}{\sqrt{|\omega|}},\ F^{-1}\{f(\omega)\} = 2[H(x+1) - H(x-1)] - \frac{1}{\sqrt{2\pi x}}$

(D) $f(x) = \frac{\cos 3x}{x^2+2},\ F\{f(x)\} = \frac{\sqrt{2}\pi}{4}\left(e^{-\sqrt{2}|\omega-3|} + e^{-\sqrt{2}|\omega+3|}\right)$

(E) None of the above

11. (5%) For any vector space V,

(A) If V is finite-dimensional, then no infinite subset of V is linearly independent
(B) If V is finite-dimensional, then V is a subspace of $\mathcal{R}^n$ for some positive integer n
(C) If V is a function space, then V must be infinite-dimensional
(D) If V is infinite-dimensional, then every infinite subset of V is linearly independent
(E) None of the preceding statements are true.

12. Which of the following is true for every orthogonal $n \times n$ matrix Q? (5%)

(A) Q has eigenvalue 1
(B) $\det Q = 1$
(C) Q^T is an orthogonal matrix
(D) Q is diagonalizable
(E) None of the preceding statements are true

13. (5%) Determine which statement is true for all $n \times n$ matrices A

(A) If A has no eigenvalues, then the degree of its characteristic polynomial is zero
(B) If one of the eigenvalues of A has multiplicity greater than one, then A has fewer then n eigenvalues
(C) If one of the eigenvalues of A has multiplicity greater than one, then A has fewer than n eigenvectors
(D) A has n distinct eigenvalues
(E) None of the preceding statements are true

14. (5%) Let A be an $n \times n$ matrix. Then which of the following set is not a subspace?

(A) Col A
(B) Row A
(C) Null A
(D) rank A
(E) None of the above

15. (5%) Suppose that A is an invertible matrix. Then:

(A) $\det A^{-1} = \det A^T$
(B) $\det A^{-1} = 1/\det A$
(C) $\det A = 0$
(D) $\det A^{-1} = \det A$
(E) None of the preceding statements is true

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16. (5%) Let $A = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 1 & -1 \\ 4 & 4 & 5 \end{bmatrix}$. Which of the following is an eigenvalue of A?
(A) 0; (B) 1; (C) 2; (D) 3.
(E) None of the above.

17. (5%) Which of the following is a linear transformation?
(A) $T : \mathcal{R} \to \mathcal{R}$ where $T(x) = 4x - 1$ for any $x \in \mathcal{R}$.
(B) $T : \mathcal{R}^2 \to \mathcal{R}^2$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 4x_2 \end{bmatrix}$ for any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{R}^2$.
(C) $T : \mathcal{P} \to \mathcal{P}$ where $T(f(x)) = f(x)(x+1)$ for any polynomial $f(x) \in \mathcal{P}$.
(D) $T : \mathcal{C}^\infty(\mathcal{R}) \to \mathcal{C}^\infty(\mathcal{R})$ where $T(f(x)) = 3f''(x) + f'(x) + f(x)$ for any differentiable function $f(x) \in \mathcal{C}^\infty(\mathcal{R})$.
(E) None of the above.

18. (5%) Which of the following sets is an orthonormal basis for the designated vector space?
(A) $\left\{\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right\}$, for $\mathcal{R}^2$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$.
(B) $\left\{\frac{1}{\sqrt{\pi}}\cos x, -\frac{1}{\sqrt{\pi}}\sin x\right\}$, for $C[0, 2\pi]$ with $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.
(C) $\{1, x, x^3\}$ for $\mathcal{P}_3$ with $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.
(D) $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ for $\mathcal{R}^3$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T M \mathbf{y}$ where $M = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.
(E) None of the above.

19. (5%) Which of the following is true?
(A) For any $m \times n$ matrix A, AA^T is always invertible.
(B) For any $m \times n$ matrix A, $\text{rank}A = n$ if and only if $A\mathbf{x} = \mathbf{0}$ has only one solution.
(C) For any $m \times n$ matrix A, $0 \leq \text{rank}(A) \leq \max(n, m)$.
(D) For an $m \times n$ matrix A, $\text{rank}A = m$ if and only if $A\mathbf{x} = \mathbf{b}$ has at least a solution for all $\mathbf{b} \in \mathcal{R}^m$.
(E) None of the above.

20. (5%) Perform Gram-Schmidt Process on the subset of $\mathcal{R}^4$, $\mathcal{S} = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 4 \\ -2 \end{bmatrix}\right\}$, and obtain an orthonormal set $\mathcal{W}$ which satisfies Span $\mathcal{S}$ = Span $\mathcal{W}$. Which of the following will NOT be an element of $\mathcal{W}$?
(A) $\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$; (B) $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$; (C) $\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$; (D) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$;
(E) None of the above.