國立臺灣大學99學年度碩士班招生考試試題

1-10 題爲填充題,請依題號於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

1. If $\begin{bmatrix} 4 & 3 & 6 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, then $(a,b,c) =$__________ (5%)

2. If $\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 3 & 3 & 3 \\ 4 & 4 & 4 & 5 & 4 & 4 \\ 5 & 5 & 5 & 5 & 6 & 5 \\ 6 & 6 & 6 & 6 & 6 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$,

   then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 =$__________ (5%)

3. If $\begin{bmatrix} a & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix}\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ and $a > 1$, then $(a,b) =$ __________ (5%)

4. If $\operatorname{rank}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & a & 1 \end{bmatrix}\right) = 3$, then $a =$__________ (5%)

5. Let $A \in R^{3\times3}$ and $\{v_1, v_2, v_3\}$ be a linearly independent subset of R^3. If $Av_1 = 3v_1 + 2v_2 + v_3$, $Av_2 = v_1 + 2v_2 - v_3$, and $Av_3 = v_1 + v_2 + v_3$, then $\det(A) =$__________ (5%)

6. If $A \in R^{5\times7}$ and $\operatorname{rank}(A) = 4$, then $\operatorname{rank}(A^TA) - \operatorname{rank}(A^T)\operatorname{rank}(A) =$______ (5%)

7. Let $u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $A = I_3 + 5uu^T$, then $u^TA^{-1}u =$__________ (5%)

8. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, then there exist $a, b \in R$ such that $(I_2 - A)^{-1} = aA + bI_2$, where $(a,b) =$__________(5%)

9. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$, then $\det(A - I_5) =$__________ (5%)

10. Let $A = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}$, then the largest eigenvalue of A is______ (5%)

見背面

第 11～14 題爲選擇題，請依題號順序於 [選擇題作答區] 內作答。

第 15 題爲非選擇題，請於 [非選擇題作答區] 內作答。

11 以下敘述何者爲正確? 請清楚標示答案，不需說明。 (10%)

(a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ is an equivalence relation on $\{1, 2, 3, 4, 5\}$.
(b) There are 2^{20} different reflexive binary relations on $\{1, 2, 3, 4, 5\}$.
(c) Let R be a binary relation. Then, $\bigcup_{i=1}^{\infty} R^i$ is the reflexive transitive closure of R.
(d) Let $(K, \cdot, +)$ be a Boolean algebra. Then, $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $a + (b \cdot c) = (a+b) \cdot (a+c)$ hold for all $a, b, c \in K$.
(e) There exists a Boolean algebra $(K, \cdot, +)$, where $|K| = 6$.

12 以下敘述何者爲正確? 請清楚標示答案，不需說明。 (10%)

(a) Suppose that $(R, +, \cdot)$ is a ring. If $a \cdot b = a \cdot c$, where $a, b, c \in R$ and a is not the identity for $+$, then $b = c$.
(b) Suppose that $(R, +, \cdot)$ is a ring and $S \subset R$ is not empty. If $a+b$, $a \cdot b \in S$ for all $a, b \in S$ and $-c \in S$ for all $c \in S$, where $-c$ denotes the inverse of c under $+$, then $(S, +, \cdot)$ is also a ring.
(c) If $(R, +, \cdot)$ is a field, then $(R, +, \cdot)$ is also an integral domain.
(d) Suppose that $(G, \cdot)$ and $(H, *)$ are two groups. A mapping $f: G \to H$ is a group isomorphism, if $f(a \cdot b) = f(a) * f(b)$ for all $a, b \in G$.
(e) If $(G, \cdot)$ is a group and $a \in G$, then $(\{a^i \mid i \in Z\}, \cdot)$ is also a group, where Z is the set of integers.

13 以下敘述何者爲正確? 請清楚標示答案，不需說明。 (10%)

(a) There are 40 different ways to store 1, 2, 3, 4, 5 in $A[1:5]$ (each number stored in a memory location) so that $A[i] \neq i$ for all $1 \leq i \leq 5$.
(b) $f(x) = x/(1-x)^2$ is the ordinary generating function for 0, 1, 2, 3, 4, ….
(c) There are 105 different integer solutions for $x_1 + x_2 + x_3 + x_4 = 24$, where $3 \leq x_i \leq 8$ for $i = 1, 2, 3, 4$.
(d) There are $(5^8 - 3^8)/2$ different ways to distribute 8 different objects $O_1, O_2, \ldots, O_8$ to five different boxes $B_1, B_2, \ldots, B_5$ provided an even number of objects are distributed to B_5.
(e) Let a_n be the number of different quaternary (i.e., $\{0, 1, 2, 3\}$) sequences of length n that have an even number of 1's, where $n \geq 2$. Then, $a_n = 2a_{n-1} + 4^{n-1}$.

14 以下敘述何者爲正確? 請清楚標示答案，不需說明。 (10%)

(In the following, $G=(V, E)$ is a simple, undirected, and weighted graph, where $V=\{v_1, v_2, \ldots, v_n\}$. Also, let d_i be the degree of v_i, where $1 \le i \le n$.)

(a) $\sum_{i=1}^{n} d_i$ is even.

(b) The connected components of G can be determined, as a consequence of executing a breadth-first search on G.

(c) A spanning tree of G can result, as a consequence of executing a breadth-first search on G.

(d) If all edge costs of G are distinct, then G has a unique minimum spanning tree.

(e) A clique of G is a complete subgraph of G.

15 Let $G=(V, E)$ be a simple undirected graph and δ be the minimum vertex degree of G, where $|V| \ge 3$ and $\delta \ge |V|/2$ are assumed. The following is a proof of G having a Hamiltonian cycle.

Suppose to the contrary that G contains no Hamiltonian cycle. Further, we assume, without loss of generality, that G is a maximal nonhamiltonian simple graph.
Let $(u, v) \notin E$. So, $G+(u, v)$, the resulting graph by augmenting G with (u, v), has a Hamiltonian cycle, denoted by $(u=)\ v_1, v_2, \ldots, v_{|V|}\ (=v)$.
Set $S=\{v_i \mid (u, v_{i+1}) \in E \text{ and } 1 \le i \le |V|-1\}$ and $T=\{v_i \mid (v_i, v) \in E \text{ and } 1 \le i \le |V|-1\}$.
Then, $|S \cup T| < |V|$ and $|S \cap T| = 0$, which implies $d_u + d_v = |S| + |T| = |S \cup T| + |S \cap T| < |V|$ (d_u and d_v are the degrees of u and v, respectively), a contradiction to $\delta \ge |V|/2$.

(a) Why $|S \cup T| < |V|$? (5%)

(b) Why $|S \cap T| = 0$? (5%)