國立臺灣大學99學年度碩士班招生考試試題

※ 注意：請於試卷上依序作答，並應註明作答之大題及小題題號。

In the following, the term "ring" means a ring with a multiplicative identity, denoted by 1; any homomorphism between two rings is assumed to send the multiplicative identity (of the source) to the multiplicative identity (of the target).

There are six problems (I) - (VI) in total; some problems contain sub-problems, indexed by (1), (2), etc.

(I) Multiple choices: For each of the following problems (1) - (4), pick one appropriate answer from the corresponding options.

(1) [5 points] Let p be a prime number. Up to isomorphism, how many groups with p^2 elements are there? Or equivalently, what is the number of isomorphism classes of groups, which consist of exactly p^2 elements?

(A) 1.

(B) 2.

(C) The number depends on the prime p.

(D) Non of the above.

(2) [5 points] Up to isomorphism, how many finite fields of 2010 elements are there?

(A) 1.

(B) 2.

(C) 3.

(D) Non of the above.

(3) [5 points] Up to isomorphism, how many vector spaces of dimension 2010 over a field F are there?

(A) 1.

(B) 2.

(C) The number depends on the field F.

(D) Non of the above.

(4) [5 points] Let R be a ring and I, J be two ideals of R. Is it true that the union $I \cup J$ is an ideal of R?

(A) Yes.

(B) No in general.

(II) [15 points] A commutative ring R is called an *integral domain* if, for any $x, y \in R$, the equality $xy = 0$ implies that either $x = 0$ or $y = 0$. Now let R be an integral domain which contains only finitely many elements. Show that R is a field.

(III) [16 points] Let $\mathbb{F}$ be a field of 169 elements; let $\mathrm{Aut}(\mathbb{F})$ be the group (under composition) of all field automorphisms of $\mathbb{F}$.

(1) Show that $\mathrm{Aut}(\mathbb{F})$ is abelian.

(2) Decompose $\mathrm{Aut}(\mathbb{F})$ into a direct sum of primary cyclic groups, i.e. find the numbers $r, e_1, \cdots, e_r$ and the primes $p_1, \cdots, p_r$ such that

$$\mathrm{Aut}(\mathbb{F}) \cong C(p_1^{e_1}) \oplus \cdots \oplus C(p_r^{e_r}),$$

where each $C(p_i^{e_i})$ is a cyclic group of order $p_i^{e_i}$.

(IV) [15 points] Let R be a commutative ring such that R has a unique maximal ideal M. Show that for any $x \in R \setminus M$, there exists $y \in R$ such that $xy = 1$.

見背面

(V) [16 points] Let G be a group of order 2010.

(1) Show that there is a unique normal subgroup of order 67 inside G.

(2) Suppose that there is a cyclic subgroup of order 30 inside G. Classify all such groups G, up to isomorphism.

(VI) [18 points] Let F be a field and $A = F[x]/(x^3 - 2)$ be the ring defined by the quotient of the polynomial ring $F[x]$ by the principle ideal generated by $x^3 - 2$. Compute the set of ring homomorphisms from A to A itself in each of the following three cases:

(1) $F = \mathbb{Q}$, the field of rational numbers.

(2) $F = \mathbb{Q}(\omega)$, where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a primitive third root of unit. ($F \cong \mathbb{Q}[z]/(z^2+z+1)$.)

(3) $F = \mathbb{F}_5$, the finite field of five elements.