Notations: $\mathbb{R}$, the real numbers; $\mathbb{C}$, the complex numbers;
$M_n(K)$, the set of all $n \times n$ matrices with entries in a field K.

(1) (10 %) Let V be a finite-dimensional vector space, U and W be subspaces of V. Suppose that $\dim(U+W) = \dim(U \cap W) + 1$. Show that either $U \subseteq W$ or $W \subseteq U$.

(2) (10 %) Let $A, B \in M_n(K)$ be two matrices, K a field. Prove that, if $AB = A + B$, then $AB = BA$.

(3) (20 %) Consider the system of linear equations

$$\begin{aligned} x + (\lambda - 1)y + (\lambda - 2)z &= 2 \\ \lambda x + (2\lambda - 2)y + (\lambda - 2)z &= \lambda + 1 \\ (\lambda^2 - 2\lambda)x + \lambda z &= -2\lambda. \end{aligned}$$

Find the real values of λ such that this system of equations satisfies the following condition.
(a) The system has no solution.
(b) The system has a unique solution.
(c) The set of the solutions of the equations is a line.
(d) The set of the solutions of the equations is a plane.

(4) (15 %) Let $\{e_1, e_2, e_3\}$ be the standard basis of the Euclidean space $\mathbb{R}^3$. Find an orientation-preserving orthogonal transformation T such that Te_1 is a unit vector normal to the plane $x + y - 1 = 0$ and Te_3 is a unit vector normal to the plane $x - y + 1 = 0$.

(5) (15 %) Let $A = (a_{ij}), B = (a_{ij}b^{i-j}) \in M_n(K)$ where $b \neq 0$. Express the determinant of B in terms of $\det A$ and b.

(6) (15 %) Consider the quadratic form

$$F(x,y,z,w) = \lambda(x^2 + y^2 + z^2 + w^2) + 2xy - 2yz + 2xz.$$

Find the real values of λ such that
(a) the quadratic form is positive definite.
(b) the quadratic form is negative definite.

(7) (15 %) Let S and T be linear transformations on a finite-dimensional vector space over $\mathbb{C}$. Suppose $ST = TS$. Show that they have a common eigenvector.

試題隨卷繳回