國立臺灣大學 113 學年度碩士班招生考試試題

1. Let $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Express $\frac{\partial z}{\partial x}$ in terms of r, θ. (10 points)

2. Let $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 7 & 3 & 2 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 2 & 7 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Denote the eigenvalues of A as $\lambda_1, \lambda_2, \lambda_3$ and λ_4. Determine the values of $\prod_{k=1}^{4} \lambda_k$ and $\sum_{k=1}^{4} \lambda_k$. (10 points)

3. Evaluate the following:

   (i) $\lim_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\mathrm{d}t$, (5 points)

   (ii) $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$, (5 points)

   (iii) $\lim_{x\to 0} \frac{1}{x} \int_0^x (1 - \tan t)^{1/t}\mathrm{d}t$, (5 points)

   (iv) $\int_0^\infty x^n e^{-x}\mathrm{d}x$ where n is a positive integer. (5 points)

4. Let V be the subspace of $\mathbb{R}^4$ generated by $\boldsymbol{v}_1 = [1, 0, 1, 0]^\top$, $\boldsymbol{v}_2 = [1, 1, 1, 0]^\top$, $\boldsymbol{v}_3 = [1, -1, 0, 1]^\top$.

   (i) Use Gram-Schmidt process to find an orthonormal basis. (10 points)

   (ii) Find the projection of $[2, 0, -1, 1]^\top$ on V. (5 points)

5. Find the local maximum/minimum values and saddle point(s) of the function

$$f(x, y) = x^2y - xy^2 + xy - y^2. \quad \text{(15 points)}$$

6. Let P_3 be the space of cubic functions (polynomials of order at most 3) with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x,$$

   and let W be the subspace spanned by $\{x^2, x^3\}$.

   (i) Give a general form of an element in P_3. (2 points)

   (ii) Determine the dimension of $W^\perp$, the orthogonal complement of W. (3 points)

   (iii) Find a basis for $W^\perp$. (10 points)

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7. Consider the quadratic form $f(x,y) = 3x^2 - 4xy + 3y^2$.

(i) Find a symmetric matrix A such that $f(x,y) = [x,y]A\begin{bmatrix} x \\ y \end{bmatrix}$. (5 points)

(ii) Find a matrix Q such that the transformation $\begin{bmatrix} x \\ y \end{bmatrix} = Q\begin{bmatrix} s \\ t \end{bmatrix}$ satisfies

$$f(x,y) = f\left(Q\begin{bmatrix} s \\ t \end{bmatrix}\right) = s^2 + 5t^2.$$ (10 points)

試題隨卷繳回